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THE BLOWHARD PROBLEM–INVISCID FLOWS WITH SURFACE INJECTION

J. D. Cole and J. Aroesty

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The RAND Corporation
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2. Bodies — Boundary layer -

3. " -- Injection

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PREFACE

Under certain conditions of flow near a slender body, when surface injection rates are large and are not dependent on convective heating, it may be possible to "blow off" the boundary layer. The flow field of such a body can then be investigated using an inviscid flow model, if the resulting shear layer is sufficiently thin.

This Memorandum considers a special class of such problems called "Blowhard Problems," where the injectant layer is thicker than the shear layer, but thin enough to preserve the slenderness of the effective body produced by the injection. High mass-transfer rates might be achieved by transpiration through porous walls or by ablation due to radiation heating. The results of this Memorandum should be useful in the interpretation of wind tunnel tests and in the prediction of aerodynamic or re-entry performance under conditions of high mass transfer.

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ABSTRACT

A simple model is presented for the analysis of inviscid flow fields over slender bodies accompanied by high rates of surface mass transfer. This model assumes a thin, inviscid injectant layer, which is separated from the outer flow by a contact discontinuity. Inviscid boundary-layer equations are shown to be applicable within the injectant layer, and the pressure gradient is found by matching pressure and flow direction at the dividing streamline. Using stream-function variables, an Abel integral equation is derived, and the inverse problem of determining the injection distribution from the shape of the dividing streamline is solved analytically for supersonic and hypersonic flow over flat plates and cones.

CONTENTS

I. INTRODUCTION*

The problem considered in this Memorandum is the construction of a model of flow past bodies with high rates of mass addition or hard blowing at the surface. Blowing is considered "hard" when the velocity component normal to the wall is an order of magnitude larger than that usually expected in a boundary layer with no blowing. Under such circumstances the boundary layer can be expected to blow off the body and become a free shear layer separating the blown flow from the free-stream flow. As a limiting case corresponding to high Reynolds numbers, this free shear layer can be regarded as a slipstream, and the inviscid flow outside the slipstream can be matched to the inviscid flow within it. The flow between the slipstream and the wall is essentially rotational. Under various simplifying assumptions, solutions can be obtained for this inner flow layer and matched to suitable external flows. In this section we present some details of the general concepts employed. In Section II the basic approximation and the solution are given for the simplest case. Additional special cases appear in Sections III and IV, and the general conclusions are presented in Section V.

It is well known that the structure of both turbulent and laminar boundary layers can be changed dramatically by fluid injection from the boundary. Injection reduces skin friction and heat transfer, increases boundary-layer thickness, and destabilizes the laminar boundary layer. Theoretical evidence(1) indicates that the laminar boundary layer on a flat plate "blows off" the wall when the blowing rate is such that

$$\frac{\rho_w v_w}{\rho_\infty U_\infty} \sim \left(\frac{\nu_\infty}{U_\infty x}\right)^{1/2}$$

Injection rates higher than this cannot be treated by boundary-layer theory.

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*Some highlights of this Memorandum were presented orally at the Fluid Dynamics Division of the American Physical Society Meeting in Hawaii, September 1965.

Some experimental evidence[2] indicates that the low-speed turbulent boundary layer on a flat plate "blows off" at an injection rate $\rho_w v_w/\rho_\infty U_\infty$ of approximately .02. Within classical flat-plate laminar boundary-layer theory, blow-off appears as a singularity; it is not possible to obtain solutions to the boundary-layer equations that satisfy the given boundary conditions beyond the blow-off point. Reference 3 contains a discussion of the effects of mass injection on incompressible boundary layers and an investigation of the blow-off singularity on a flat-plate boundary layer with a constant injection rate.

Pretsch[4] examined the Falkner-Skan flows for asymptotically high injection rates and favorable pressure gradients. He demonstrated that solutions are possible for arbitrarily large injection rates, and showed that skin friction approaches zero as $\beta/(-f_w)$, where (f_w) is the wall injection rate in the appropriate similarity variable and β is the pressure-gradient parameter. Pretsch also showed that neglecting the viscous terms in the equation of motion for high injection rates produces an inviscid balance between inertia and pressure gradient. The resulting inviscid equation, solved by Pretsch, has a discontinuity in vorticity at the dividing streamline. In addition, for certain values of β, the shear becomes infinite at the dividing streamline, making it necessary to introduce a viscous shear layer at the dividing streamline to resolve this singularity. An extension of Pretsch's original inviscid analysis to the details of the shear layer is given in Refs. 5 and 6.

More recently, Libby[7] and Vinokur[8] have discussed the problem of the stagnation region of a sphere in hypersonic flow when mass is injected from the surface of the sphere. Their papers treat the problem of inviscid flow with mass addition within the framework of hypersonic, constant-density, blunt-body flow. Their solutions are characterized by two regions separated by a surface of discontinuity -- the dividing streamline. The inner region consists of mass injected from the wall, and the outer region of fluid that has passed through the bow shock wave. The inner flow may be rotational or irrotational, depending on the manner of fluid injection, whereas the outer flow is rotational because of the curvature of the shock wave. Since there is a stagnation point in the flow through which the dividing streamline

passes, the total pressure, static pressure, and flow direction are continuous as the discontinuity is approached from either the inner injection zone or the outer shock layer. Vinokur found that the thicknesses of the shock layer and the injection layer are functions of the nondimensional injection momentum $\rho_w V_w^2/(\rho_\infty V_\infty^2)$ alone.

Ting[9] has recently considered a model of distributed injection on a flat plate, with the restriction that the inner flow is incompressible and potential and the outer flow is linearized.

In this Memorandum we consider the effects of mass transfer on flows that, in the absence of mass transfer or viscous effects, would have a pressure gradient of zero along the surface. Examples of such flows are supersonic wedge and cone flows. For such flows, high injection rates could result in blow-off, making it necessary to re-evaluate the usual procedures of boundary layer theory.

According to boundary-layer theory (as some typical Reynolds number approaches infinity), the inviscid flow around the object is first calculated, establishing a velocity and a pressure field for the boundary layer. This inviscid flow typically slips along the solid surface, and a viscous boundary layer along the surface must be introduced to satisfy a no-slip condition. In this theory, blowing must vanish as $Re \to \infty$. Higher-order interaction effects, such as flow due to displacement thickness, can then be calculated. When the blowing rate is sufficiently high this approximation scheme may not be suitable. The first step in a more appropriate calculation scheme is to find the inviscid flow past the object with blowing. This flow contains a dividing streamline, off the body surface, which separates the injected fluid from the oncoming stream (see Fig. 1). In inviscid theory, there is a jump in tangential velocity across the separating streamline. The flow in the injection layer is generally rotational, but may to the first approximation be considered inviscid. The viscous boundary layer becomes a free shear layer which resolves the discontinuity at the dividing streamline.

In the following sections we consider a simple case of this type of flow, in which blowing produces only a thin region. An assessment of orders of magnitude shows that the flow in the blown layer can be

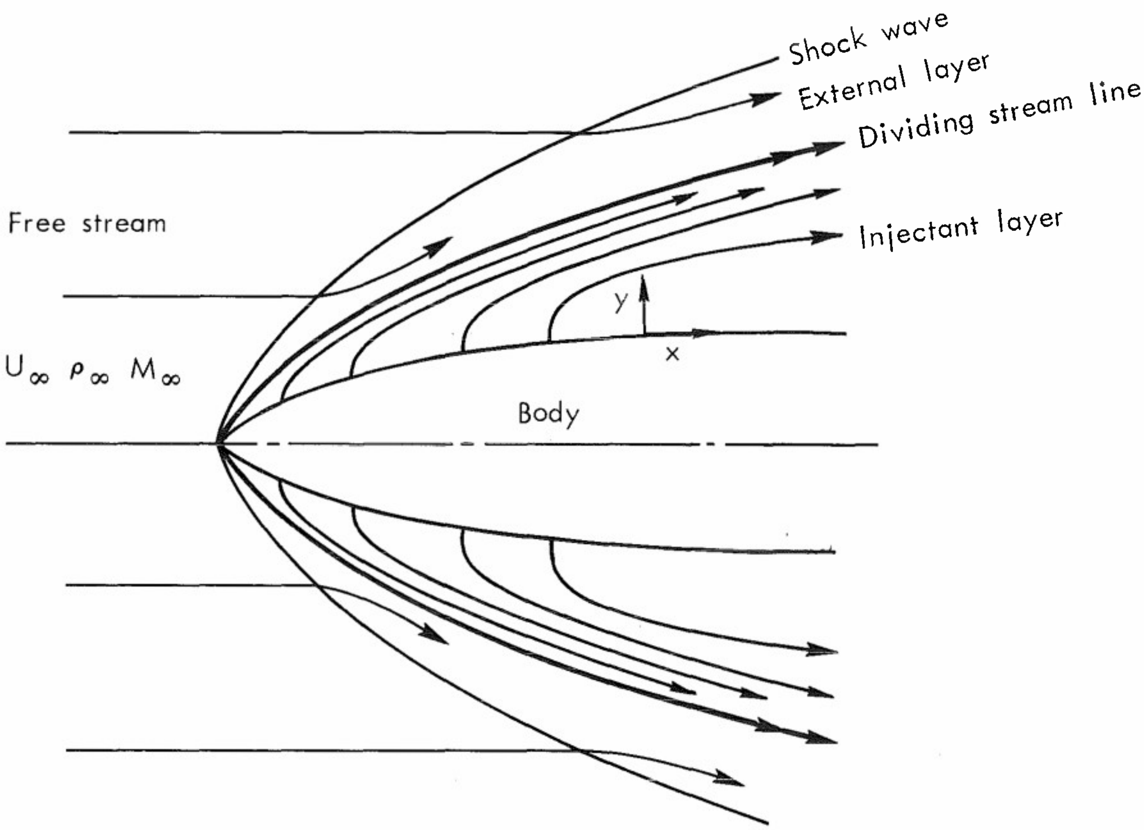


Fig. 1 -- Injection into supersonic flow (inviscid model).

described by inviscid boundary-layer equations. These equations are simple enough to provide an explicit relation between the distribution of blowing velocity, the distribution of pressure, and the shape of the dividing streamline. If, in addition, there is some simple relation between shape and pressure, as for example in linearized supersonic flow, then explicit values of all the quantities can be found.

This simple theory should be applicable whether the shear layer is laminar or turbulent, so long as it is thin relative to the region of blown flow.

It should also be noted that the case of strong suction is completely different from strong blowing; the boundary layer always occurs at the wall. However, since the boundary layer does not occur on a streamline of the inviscid flow, the boundary-layer equations are ordinary differential equations and the boundary-layer thickness is $0(1/Re)$.

II. INCOMPRESSIBLE INNER LAYER

Consider flow past a flate plate ($x > 0$) in a uniform stream with velocity U. Assume as boundary conditions a distribution of blowing velocity $q_y(x,0)$ along the plate and a tangential velocity $q_x(x,0) = 0$. The latter condition is not necessary but is believed to correspond to the correct limit of the viscous flow equations unless special precautions are taken with the blowing. These boundary conditions introduce rotation into the injection layer. The additional boundary condition can be accommodated, since the location of the dividing streamline is not known in advance but must be found as part of the solution to the problem. The conditions to be satisfied across the separating streamline are continuity of pressure and direction of flow.

Let the dividing streamline be represented by

$$y = \delta S(x) \tag{2.1}$$

where $S(0) = 0$, $S(1) = 1$, and δ, the basic small parameter of the problem, is the thickness of the blown layer at a unit distance downstream of the nose. The thickness of the layer tends to zero as $\delta \to 0$, corresponding to the disappearance of both blowing and the pressure and tangential velocity perturbations. In order to obtain approximate equations valid within the thin layer, distances normal to the wall must be measured in terms of the thickness of the layer. That is, a coordinate

$$\tilde{y} = \frac{y}{\delta} \tag{2.2}$$

is held fixed as $\delta \to 0$. Assume now that the flow in the thin layer is incompressible (more precise conditions for the validity of this assumption will be given later). Thus the following forms of asymptotic expansions in δ can be assumed for the flow quantities in the thin layer:

$$\frac{q_x(x,y;\delta)}{U} = \alpha(\delta)u(x,\tilde{y}) + \ldots \tag{2.3}$$

$$\frac{q_y}{U}(x,y;\delta) \quad = \quad \beta(\delta)v(x,\tilde{y}) + \ldots \tag{2.4}$$

$$\frac{p(x,y;\delta) - p_\infty}{\rho_\infty U^2} \quad = \quad \epsilon(\delta)\tilde{p}(x,\tilde{y}) + \ldots \tag{2.5}$$

The continuity equation is nontrivial for $\beta = \delta\alpha$, and the tangential momentum is balanced between inertia and pressure gradient for $\epsilon = \alpha^2$:

$$\left.\begin{aligned} \beta &= \alpha\delta \\ \epsilon &= \alpha^2 \end{aligned}\right\} \tag{2.6}$$

The resulting approximate equations are

$$\text{continuity:} \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial \tilde{y}} \quad = \quad 0 \tag{2.7}$$

$$\text{x-momentum:} \qquad u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial \tilde{y}} \quad = \quad -\sigma\frac{\partial \tilde{p}}{\partial x} \tag{2.8}$$

$$\text{y-momentum:} \qquad 0 \quad = \quad -\sigma\frac{\partial \tilde{p}}{\partial \tilde{y}} \tag{2.9}$$

where

σ = density ratio (of order one) = ρ_∞/ρ_w.

Thus we obtain essentially the inviscid version of the Prandtl boundary-layer equations. Hence the results above still apply to a curved surface with an arc-length x and a normal distance y, so long as the radius of curvature is larger than $O(\delta)$ in characteristic units.

A consequence is that the pressure in the layer $\tilde{p}$ is only a function of x:

$$\tilde{p} = P(x) \tag{2.10}$$

and we obtain the further simplified equations

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial \tilde{y}} = 0 \tag{2.11}$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial \tilde{y}} = -\sigma \frac{dP}{dx} \tag{2.12}$$

We can now estimate the Mach number M_ℓ in the inner layer by using

$$M_\ell^2 \sim \frac{U^2 \alpha^2(\delta) \rho_\infty}{\gamma \sigma p_\infty [1 + \ldots]} \sim M_\infty^2 \alpha^2(\delta) \tag{2.13}$$

We see that if the external Mach number is fixed as $\delta \rightarrow 0$, then the flow is incompressible, although compressibility must be taken into account in the hypersonic case ($M_\infty \rightarrow \infty$). We can also estimate the neglected viscous-stress terms in Eq. (2.8) by calculating the ratio

$\frac{\sigma \nu_\infty \ell}{\alpha \delta^2 U}$

$$\frac{\frac{\mu}{\rho_w} \frac{\partial^2 q_x}{\partial y^2}}{q_x \frac{\partial q_x}{\partial x}} \sim \frac{\sigma \nu_\infty}{\alpha(\delta) \delta^2 \ell} \sim \frac{\sigma}{(\delta^2 \alpha)(Re)} \tag{2.14}$$

We assume that the Reynolds number based on the characteristic length ($\ell = 1$) is large enough to make the ratio in Eq. (2.14) much less than one. For the theory to apply, Re must be regarded as $Re(\delta) \gg 1/(\delta^2 \alpha)$. Thus for a given Reynolds number the theory is consistent when

$$\frac{\sigma}{Re} \ll \alpha \delta^2 \ll 1 \tag{2.15}$$

That is, the layer must be thin, but not too thin.

The system of Eqs. (2.11) and (2.12) is now to be solved with the boundary conditions

$$v(x,0) = v_w(x), \qquad u(x,0) = 0 \tag{2.16}$$

It is convenient at first to regard $P(x)$ as known and to see if some relation can be found among $v_w(x)$, $P(x)$, and $S(x)$. This is most easily done if we introduce a new coordinate that is constant along a streamline in the blown layer. Let the stream function $\psi(x,\tilde{y})$ be defined by

$$u(x,\tilde{y}) = \frac{\partial\psi}{\partial\tilde{y}}, \qquad v(x,\tilde{y}) = -\frac{\partial\psi}{\partial x} \tag{2.17}$$

Then on the surface ($\tilde{y} = 0$)

$$v_w(x) = -\frac{\partial\psi}{\partial x}(x,0)$$

or

$$\psi_w(x) \equiv \psi(x,0) = -\int_0^x v_w(\xi)\,d\xi \tag{2.18}$$

Let $x^*(\psi)$ be the x-coordinate where a streamline enters the flow at $\tilde{y} = 0$; that is, the inverse function of Eq. (2.18) (see Fig. 2).

Using v_w to represent the velocity at the plate, we have

$$dx^* = -\frac{d\psi}{v_w(\psi)}$$

or

$$x^*(\psi) = -\int_0^\psi \frac{d\Psi}{v_w(\Psi)} \tag{2.19}$$

The problem can be formulated in the (x^*,x) plane (Fig. 2) where the surface $\tilde{y} = 0$ is $x^* = x$, the separating streamline is $x^* = 0$, and $0 \le x^* \le x$.

We apply the following transformations from $(x,\tilde{y})$ to (x,x^*):

$$\frac{\partial}{\partial x} \to \frac{\partial}{\partial x} + \frac{\partial x^*}{\partial x}\frac{\partial}{\partial x^*} = \frac{\partial}{\partial x} + v\frac{\partial x^*}{\partial \psi}\frac{\partial}{\partial x^*} = \frac{\partial}{\partial x} + \frac{v(x,x^*)}{v_w(x^*)}\frac{\partial}{\partial x^*} \tag{2.20}$$

$$\frac{\partial}{\partial \tilde{y}} \to \frac{\partial x^*}{\partial \tilde{y}}\frac{\partial}{\partial x^*} = -\frac{u(x,x^*)}{v_w(x^*)}\frac{\partial}{\partial x^*} \tag{2.21}$$

$$u\frac{\partial}{\partial x} + v\frac{\partial}{\partial \tilde{y}} \to u(x,x^*)\frac{\partial}{\partial x} \tag{2.22}$$

Thus the tangential momentum equation (2.12) becomes, in the new coordinate,

$$u\frac{\partial u}{\partial x} = -\sigma\frac{dP}{dx} \tag{2.23}$$

This can be integrated to give the approximate version of the Bernoulli equation if the no-slip condition

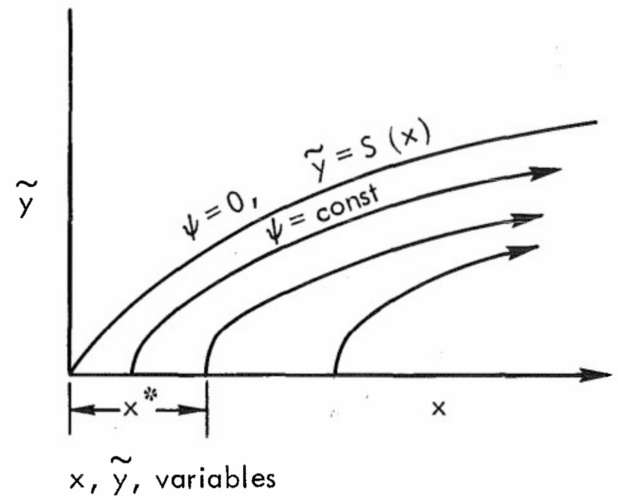
$$u(x^*,x^*) = 0 \tag{2.24}$$

is taken into account. This approximate Bernoulli equation is:

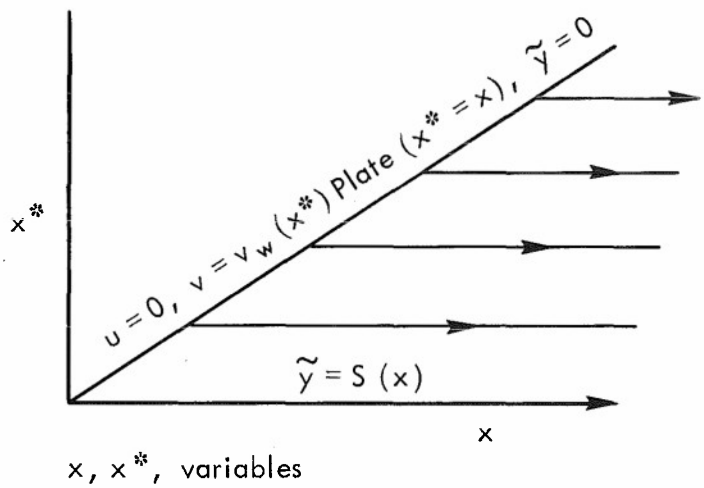
$$\frac{1}{2}u^2(x,x^*) + \sigma P(x) = \sigma P(x^*) \tag{2.25}$$

The horizontal component of the flow is

$$u(x,x^*) = (2\sigma)^{1/2}\left(P(x^*) - P(x)\right)^{1/2} \tag{2.26}$$



(a)



(b)

Fig. 2 -- Coordinate transformations and boundary conditions.

✱ This result implies that the solution is valid only for a favorable pressure gradient with $P(x^*) > P(x)$.

Next the shape of the dividing streamline is calculated by integration across the streamlines.

For x = constant,

$$d\tilde{y} = \frac{d\psi}{u} = \frac{d\psi}{dx^*}\frac{dx^*}{u} = -\frac{v_w(x^*)}{u(x,x^*)}\,dx^* \tag{2.27}$$

Thus, integrating Eq. (2.27), with the boundary condition that the dividing streamline originates at the nose [$\tilde{y} = S(x)$ for $x^* = 0$], we obtain

$$\tilde{y}(x,x^*) = S(x) - \int_0^{x^*} \frac{v_w(\xi)}{u(x,\xi)}\,d\xi \tag{2.28}$$

Using the Bernoulli equation (2.26), we replace the tangential velocity in the last expression by its equivalent in terms of pressure:

$$\tilde{y}(x,x^*) = S(x) - \frac{1}{(2\sigma)^{1/2}} \int_0^{x^*} \frac{v_w(\xi)}{\left(P(\xi) - P(x)\right)^{1/2}}\,d\xi \tag{2.29}$$

Finally, this last expression can be evaluated on the surface where $x^* = x$ and $\tilde{y} = 0$ to provide a relationship among $S(x)$, $v_w(x)$, and $P(x)$:

$$S(x) = \frac{1}{(2\sigma)^{1/2}} \int_0^{x} \frac{v_w(\xi)}{\left(P(\xi) - P(x)\right)^{1/2}}\,d\xi \tag{2.30}$$

In general, the relation between the pressure and the shape of the dividing streamline depends on the nature of the external flow. It is therefore convenient to express $v_w(x)$ as an integral over P,S, since Eq. (2.30) is an integral equation of Abel type for $v_w(\xi)$ (see Appendix for details):

$$v_w(x) = -\frac{1}{\pi}(2\sigma)^{1/2}\frac{dP}{dx}\int_0^x \frac{S'(\xi)}{\left(P(\xi) - P(x)\right)^{1/2}}\,d\xi \tag{2.31}$$

Equation (2.31) is the basic answer to our problem, and provides the desired information in an inverse form. The blowing distribution $v_w(x)$ can be found for a given $P(x)$, $S(x)$.

We shall now apply this theory concretely by specifying the outer flow and its perturbations. First, we can fix the order of magnitude α,β,ϵ of the perturbation by constructing a model of the outer flow. For example, consider flow past a flat plate and conditions for which linearized subsonic or supersonic theory is valid; that is,

$$\frac{\delta}{\left(|M_\infty^2 - 1|\right)^{1/2}} \ll 1 \tag{2.32}$$

Then it is well known that the pressure perturbations in the outer flow are of the same order as the body thickness or shape; that is,

$$\begin{aligned} \epsilon(\delta) &= \delta \\ \alpha(\delta) &= \delta^{1/2} \\ \beta(\delta) &= \delta^{3/2} \end{aligned} \tag{2.33}$$

The last equation relates δ to the order of magnitude of the blowing velocity:

$$\delta \sim \left(\frac{q_y(x,0)}{U}\right)^{2/3} \tag{2.34}$$

To ensure the self-consistency of the theory, the following more precise estimates of Reynolds number and blowing velocity can now be obtained from Eq. (2.15):

$$\left(\frac{\sigma}{Re}\right)^{2/5} \ll \delta \ll 1$$

or

$$\left(\frac{\sigma}{Re}\right)^{3/5} \ll \frac{q_y(x,0)}{U} \ll 1$$

If this theory is to supersede classical boundary-layer theory (for $\sigma = 1$), it is necessary that $q_y/U > 1/(Re)^{1/2}$. This condition ensures that the thickness of the blown layer, δ, is greater than a characteristic thickness of the shear layer, $1/(Re)^{1/2}$

Thus, for linearized external flow, the basic expansion [(Eqs. (2.3) through (2.5)] becomes

$$\frac{q_x}{U}(x,y;\delta) = \delta^{1/2} u(x,\tilde{y}) + \ldots \tag{2.35}$$

$$\frac{q_y}{U}(x,y;\delta) = \delta^{3/2} v(x,\tilde{y}) + \ldots \tag{2.36}$$

$$\frac{p(x,y;\delta) - p_\infty}{\rho_\infty U^2} = \delta P(x) + \ldots \tag{2.37}$$

The simplest results are obtained for linearized supersonic ($M_\infty > 1$) external flow, where the following relation between pressure perturbation and local slope is valid:

$$P(x) = \frac{S'(x)}{\left(M_\infty^2 - 1\right)^{1/2}} \tag{2.38}$$

Thus, the blowing velocity and shape of the dividing streamline can be expressed in terms of a given pressure distribution from Eq. (2.31):

$$v_w(x) = -\frac{1}{\pi}\left((M_\infty^2 - 1)2\sigma\right)^{1/2} \frac{dP}{dx}\int_0^x \frac{P(\xi)}{\left(P(\xi) - P(x)\right)^{1/2}}\, d\xi \qquad (2.39)$$

$$S(x) = \left(M_\infty^2 - 1\right)^{1/2} \int_0^x P(\xi)\, d\xi \qquad (2.40)$$

We next apply the foregoing method to some specific examples.

Example 1 -- Ogive Nose

Let the pressure distribution be given by

$$P(x) = a_0 - a_1 x \qquad 0 < x < 1 \qquad (2.41)$$

Then Eq. (2.39) gives

$$v_w(x) = \frac{a_1}{\pi}\left(2\sigma(M_\infty^2 - 1)\right)^{1/2} \int_0^x \frac{a_0 - a_1\xi}{\left(a_1(x - \xi)\right)^{1/2}}\, d\xi$$

$$= \frac{1}{\pi}\left(2\sigma(M_\infty^2 - 1)\right)^{1/2} (a_1)^{1/2} \left\{2a_0 x^{1/2} - \frac{4}{3} a_1 x^{3/2}\right\} \qquad (2.42)$$

and Eq. (2.40) gives

$$S(x) = \left(M_\infty^2 - 1\right)^{1/2} \left\{a_0 x - \frac{1}{2} a_1 x^2\right\} \qquad (2.43)$$

The normalization $S(1) = 1$ gives the following relation between a_1 and a_0:

$$a_0 = \frac{1}{2} a_1 + \frac{1}{\left(M_\infty^2 - 1\right)^{1/2}} \qquad (2.44)$$

However, the condition of favorable pressure gradient requires that

$$a_1 > 0$$

so that the solution is only valid for

$$a_0 > \frac{1}{\left(M_\infty^2 - 1\right)^{1/2}} \tag{2.45}$$

It is seen from this example that an ogive-like layer with a constant slope $\delta(M_\infty^2 - 1)^{1/2}$ a_0 at the nose is produced by blowing which causes $\sim (x)^{1/2}$ to vanish as $x \to 0$; however, it is the curvature of the ogival layer near the nose which determines the required injection distribution as $x \to 0$. For example, for S to be $Ax - Bx^{n+1}$, and P to be $A(n + 1)Bx^n$, it is necessary that $v_w \sim x^{n/2}$ as $x \to 0$. The injection velocity must vanish as $x \to 0$ in order to maintain the finite slope of the injection layer.

Example 2 -- Power-Law Nose

Let

$$P(x) = \frac{1 - n}{\left(M_\infty^2 - 1\right)^{1/2}} x^{-n} \tag{2.46}$$

$$S(x) = x^{1-n} \qquad n < 1 \tag{2.47}$$

Then

$$\int_0^x \frac{\xi^{-n}}{\left(\xi^{-n} - x^{-n}\right)^{1/2}} d\xi = \frac{(\pi)^{1/2}}{n} \frac{\Gamma\left(\frac{1}{n} - \frac{1}{2}\right)}{\Gamma\left(\frac{1}{n}\right)} x^{1-n/2} \qquad 0 < n < 2 \tag{2.48}$$

so that

$$v_w(x) = \left(\frac{2\sigma}{\pi}\right)^{1/2} \frac{1}{\left(M_\infty^2 - 1\right)^{1/4}} \frac{\Gamma\left(\frac{1}{n} - \frac{1}{2}\right)}{\Gamma\left(\frac{1}{n}\right)} (1 - n)^{3/2} x^{-3n/2} \qquad (2.49)$$

There is a finite rate of mass addition only for $n < 2/3$. The often-used case where $v_w \sim x^{-1/2}$ corresponds here to $p \sim x^{-1/3}$, $S \sim x^{2/3}$. Note however, that because $P \rightarrow \infty$ as $x \rightarrow 0$, the Bernoulli integral (2.26) indicates that $u \rightarrow \infty$ on the dividing streamline. The solution is probably not valid near the dividing streamline for these cases, and the calculation of a free shear layer would require further discussion.

The order of magnitude of the jump in tangential velocity at the separation streamline can be calculated. Since the external velocity is $U + 0(\delta)$ and the tangential velocity in the blown layer is $0(\delta)^{1/2}$, the jump in tangential velocity $[q]$ is expressed in terms of $u(x,x^*)$ as

$$\frac{1}{U}[q]_{Tang} = 1 - (\delta)^{1/2} u(x,0) = 1 - (2\delta\sigma)^{1/2}[P(0) - P(x)] \qquad (2.50)$$

Within the context of this thin-layer theory, the vorticity is always finite and is introduced at the wall by the stagnation pressure gradient of the injectant; the skin friction coefficient is

$$- \frac{1}{\delta^{1/2} Re_\infty} \frac{\sigma^{1/2}}{v_w} \frac{\partial P}{\partial x} .$$

III. OTHER EXAMPLES OF INCOMPRESSIBLE INJECTION LAYERS

In this section we consider two further examples of inviscid flows with mass transfer at the surface, which are characterized, like the flat plate in linearized supersonic flow, by incompressible flow in the thin injection layer.

HYPERSONIC WEDGE FLOW WITH DISTRIBUTED BLOWING

The simple theory presented in Section II can be readily extended to the problem of supersonic flow past an inclined flat plate (wedge), with distributed mass injection along the surface of the wedge. We consider only the case where the injection-layer thickness δ is small compared to the original wedge height θ.

According to the theory of supersonic flow past a slightly perturbed wedge, flow quantities in the outer flow can be expressed as

$$q_x(x,y) = U_2[1 + \epsilon u(x,y)] \tag{3.1}$$

$$q_y(x,y) = U_2 \epsilon \tilde{v}(x,y) \tag{3.2}$$

$$p(y) = p_2 + \epsilon \rho_2 U_2^2 \tilde{p}(x,y) + \dots \tag{3.3}$$

where ϵ is a small parameter determined by the magnitude of the perturbation, and the subscript "2" refers to conditions behind the original shock wave.

The condition of tangential flow at the outer edge of the injectant layer is replaced by a boundary condition on the original wedge surface, and the perturbed shock jump conditions are given along the original unperturbed shock wave of the wedge. Reference 10 contains a complete discussion of this theory.

A solution of the wave equation with appropriate boundary conditions results in the following equation for the pressure:

$$p\Big(x,y(x)\Big) = p_2 + \frac{\rho_2 U_2^2}{\left(M_2^2 - 1\right)^{1/2}} \left[y'(x) + 2 \sum_{n=1}^{\infty} \lambda^n y'(k^n x)\right] \tag{3.4}$$

where

λ = reflection coefficient

k = ratio of the x-coordinate where a wave originates near the wedge to the x-coordinate where the wave intersects the wedge

$y(x)$ = dimensional perturbation measured from the wedge surface.

If this result is specialized to the hypersonic limit, $M_\infty \to \infty$, $\theta_{wedge} \to 0$, $M_\infty \theta_{wedge} \to \infty$, then

$$\lambda \to \lambda_\infty = \frac{1 - \left(\frac{\gamma}{2(\gamma - 1)}\right)^{1/2}}{1 + \left(\frac{\gamma}{2(\gamma - 1)}\right)^{1/2}} \tag{3.5a}$$

$$k \to k_\infty = \frac{1 - \left(\frac{\gamma - 1}{2\gamma}\right)^{1/2}}{1 + \left(\frac{\gamma - 1}{2\gamma}\right)^{1/2}} \tag{3.5b}$$

Equation (3.4) becomes

$$p\Big(x, \delta S(x)\Big) = (\gamma + 1)\theta^2 \times \frac{\rho_\infty U_\infty^2}{2} \left\{1 + \frac{\delta}{\theta}\left(\frac{2\gamma}{\gamma - 1}\right)^{1/2} \left[S'(x) + 2 \sum_{n=1}^{\infty} \lambda^n S'(k^n x)\right]\right\} \tag{3.6}$$

where the shape of the bounding streamline is $Y(x) = \theta(\delta/\theta)S(x)$, and $\delta/\theta \ll 1$.

<u>Inner Flow</u>

The injected fluid is assumed to leave the surface with a constant temperature T_w. This assumption is not essential to the analysis, however, and the theory remains valid for moderate variations of wall temperature.

We introduce a new coordinate $\tilde{y} = y/\delta$. Consistent orders of magnitude for flow variables in the injection layer are as follows:

$$q_x(x,y) = \left(\frac{R_0 T_w}{\mu_w}\right)^{1/2} \left(\frac{\delta}{\theta}\right)^{1/2} u(x,\tilde{y}) + \ldots \tag{3.7a}$$

$$q_y(x,y) = \left(\frac{R_0 T_w}{\mu_w}\right)^{1/2} \theta\left(\frac{\delta}{\theta}\right)^{3/2} v(x,\tilde{y}) + \ldots \tag{3.7b}$$

$$\rho(x,y) = \frac{\gamma + 1}{2} \frac{\rho_\infty U_\infty^2}{\frac{R_0 T_w}{\mu_w}} \theta^2 \bar{\rho}(x,\tilde{y}) + \ldots \tag{3.7c}$$

$$p(x,y) = \frac{\gamma + 1}{2} \rho_\infty U_\infty^2 \theta^2 \left\{1 + \frac{\delta}{\theta} p_s(x,\tilde{y}) + \ldots\right\} \tag{3.7d}$$

where R_0 is the universal gas constant and μ_w is the molecular weight of the injectant. The equations of motion then become

$$\text{x-momentum:} \qquad u \frac{\partial u}{\partial x} + v \frac{\partial y}{\partial \tilde{y}} = - \frac{\partial p_S}{\partial x} \left(\frac{1}{\bar{\rho}}\right) \tag{3.8}$$

$$\text{continuity:} \qquad \frac{\partial(u\bar{\rho})}{\partial x} + \frac{\partial(v\bar{\rho})}{\partial \tilde{y}} = 0 \tag{3.9}$$

$$\text{y-momentum:} \qquad \frac{\partial p_S}{\partial \tilde{y}} = 0 \tag{3.10}$$

If we assume a perfect gas with an isentropic exponent γ_w, then the entropy equation can be written

$$\frac{p}{\bar{\rho}^\gamma} = f(\psi)$$

If the wall temperature is constant and the injection Mach number is small, then the flow is incompressible, since $\bar{\rho} = 1 + O(\delta/\theta)$. The

boundary conditions at the wall become $u(x,0) = 0$, $v(x,0) = v_w(x)$. The boundary conditions on the dividing streamline, $\tilde{y} = S(x)$, are

$$\frac{v\big(x,S(x)\big)}{u\big(x,S(x)\big)} = S'(x) \tag{3.11}$$

and

$$p_S\big(x,S(x)\big) = \left(\frac{2\gamma}{\gamma - 1}\right)^{1/2}\left[S'(x) + 2\sum_{n=1}^{\infty} \lambda^n S'(k^n x)\right] \tag{3.12}$$

As in Section II, the equations of motion and entropy reduce to those of an inviscid, incompressible boundary layer. The density is constant, and there is no pressure gradient across the layer.

In terms of the mass flow ratio per unit area, $\dot{m} \equiv (\rho q_y)_w/\rho_\infty U_\infty$, the requirement that $\delta/\theta \ll 1$ results in the following estimate:

$$\dot{m} = \theta^3\left(\frac{\delta}{\theta}\right)^{3/2} \times \frac{U_\infty}{\left(\frac{R_0 T_w}{\mu_w}\right)^{1/2}}$$

or

$$\frac{\delta}{\theta} = \left[\frac{\dot{m}}{M_\infty \theta^3}\left(\frac{T_w}{T_\infty}\frac{\mu_\infty}{\mu_w}\right)^{1/2}\right]^{2/3} \ll 1$$

or

$$\dot{m} \ll M_\infty \theta^3\left(\frac{T_w}{T_\infty}\frac{\mu_\infty}{\mu_w}\right)^{1/2}$$

For a fixed Mach number M_∞, a fixed mass flow $\dot{m}$, a fixed wedge angle θ, and a fixed wall-to-free-stream temperature ratio T_w/T_∞, this implies that

$$\frac{\delta}{\theta} \sim \left[\left(\frac{\mu_\infty}{\mu_w}\right)^{1/2}\right]^{1/3}$$

Thus, the injection of a light gas into a heavy one produces greater induced pressure effects than does the injection of a heavy gas into a light one. For example, pressure perturbations due to the injection of helium into air are predicted to be about twice those for air into air.

The flow in the injection layer is governed by the same equations as those given earlier, and we obtain the simple Bernoulli equation:

$$\frac{u^2}{2}(x,x^*) + p_S(x,x^*) = p_S(x^*) \tag{3.13}$$

where x^* is defined in Eq. (2.19). The height of the dividing streamline is again

$$S(x) = \frac{1}{2^{1/2}} \int_0^x \frac{v_w(\xi)}{\left(p_S(\xi) - p_S(x)\right)^{1/2}} d\xi \tag{3.14}$$

where

$$p_S(x) = \left(\frac{2}{\gamma - 1}\right)^{1/2} \left\{S'(x) + 2 \sum_{n=1}^{\infty} \lambda^n S'(k^n x)\right\} \tag{3.15}$$

As before, an inverse representation of the solution is possible: $S(x)$ is assumed, the resulting pressure distribution is calculated from Eq. (3.15), and Eq. (3.14) then becomes an Abel integral equation for $v_w(x)$.

Example

If

$$S'(x) = \sum_{n=0}^{N} a_n x^n \tag{3.16}$$

then

$$p_S(x,0) = \left(\frac{2}{\gamma - 1}\right)^{1/2} \left\{ \sum_{n=0}^{N} \left[\frac{1 + \lambda_\infty k_\infty^n}{1 - \lambda_\infty k_\infty^n}\right] a_n x^n \right\} \tag{3.17}$$

If the blowing results in an ogive-like injection zone,

$$\left.\begin{aligned} S &= a_1 x - a_2 \frac{x^2}{2} \\ S' &= a_1 - a_2 x \end{aligned}\right\} \quad \text{where } a_1,\ a_2 > 0 \tag{3.18}$$

The pressure formula gives

$$p_S = \left(\frac{2}{\gamma - 1}\right)^{1/2} \left\{ \frac{1 + \lambda_\infty}{1 - \lambda_\infty} a_1 - \frac{1 + \lambda_\infty k_\infty}{1 - \lambda_\infty k_\infty} a_2 \right\} \tag{3.19}$$

and from the Appendix, $v_w(x)$ is then

$$v_w = + \frac{1}{\pi} \left[\left(\frac{2}{\gamma - 1}\right)^{1/2} \times \frac{1 + \lambda_\infty k_\infty}{1 - \lambda_\infty k_\infty} \times a_2 \times 2 \right]^{1/2} \int_0^x \frac{(a_1 - a_2 \xi)}{x - \xi} \, d\xi \tag{3.20}$$

or

$$v_w(x) = \frac{1}{\pi} \left[\left(\frac{2}{\gamma - 1}\right)^{1/2} \frac{1 + \lambda_\infty k_\infty}{1 - \lambda_\infty k_\infty} \times a_2 \times 2 \right]^{1/2} \left[x^{1/2} \times a_1 \times 2 - x^{3/2} \times a_2 \times \frac{4}{3} \right] \tag{3.21}$$

Since $S(1) = 1$, then

$$a_1 - \frac{a_2}{2} = 1 \tag{3.22}$$

or

$$a_2 = 2\left\{a_1 - 1\right\} \tag{3.23}$$

HYPERSONIC NEWTONIAN FLOW PAST A POROUS CONE

Another example of the "blowhard" problem is provided by supersonic or hypersonic flow past a porous cone when the injection rates are high. We restrict our attention to the study of hypersonic flow past a porous slender cone of angle θ, and use the Newtonian formula for the calculation of the pressure distribution on a perturbed cone. As before, the "blown" layer is assumed to be thin relative to the thickness of the original cone.

Newtonian slender-body flow corresponds to the limits $\theta \to 0$, $\gamma \to 1$, and $1/M^2\theta^2(\gamma - 1) \to 0$. The Newtonian formula, including impact and centrifugal force terms, is

$$p = \frac{\rho_\infty U_\infty^2}{2} \left[r'^2 + \frac{rr''}{2}\right] \tag{3.24}$$

where the profile shape is $r = r(x)$ and primes indicate d/dx.

If the injectant layer height at $x = 1$ is δ, then

$$r = \theta\left[x + \left(\frac{\delta}{\theta}\right) S(x)\right] \tag{3.25}$$

where $\delta/\theta \ll 1$. The initial height and the normalization condition are given by:

$$S(0) = 0, \qquad S(1) = 1 \tag{3.26}$$

This pressure formula then becomes

$$p = \frac{\rho_\infty U_\infty^2}{2} \theta^2 \left\{1 + 2\,\frac{\delta}{\theta}\, S'(x) + \left(\frac{\delta}{\theta}\right) \times \frac{S''}{2}\,(x) + 0\left(\frac{\delta}{\theta}\right)^2\right\} \tag{3.27}$$

The following orders of magnitude are appropriate for the flow variables in the inner layer:

$$q_x = \left(\frac{R_0 T_w}{\mu_w}\right)^{1/2} \frac{\delta}{\theta}\, u(x,\tilde{y}) \tag{3.28a}$$

$$q_y = \theta\left(\frac{R_0 T_w}{\mu_w}\right)^{1/2} \left(\frac{\delta}{\theta}\right)^{3/2} v(x,\tilde{y}) \tag{3.28b}$$

$$p = \frac{\rho_\infty U_\infty^2 \theta^2}{2}\left[1 + \frac{\delta}{\theta}\, p_S(x,\tilde{y}) + \ldots\right] \tag{3.28c}$$

$$\rho = \frac{\rho_\infty U_\infty \theta^2}{\dfrac{2R_0 T_w}{\mu_w}}\, \bar{\rho}(x,\tilde{y}) \tag{3.28d}$$

The equations of motion become, in boundary-layer coordinates,

$$u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} = -\,\frac{\partial p_S}{\partial x \bar{\rho}} \tag{3.29}$$

$$\frac{\partial}{\partial x}\left[r_0(x)u\right] + \frac{\partial}{\partial \tilde{y}}\left[r_0(x)v\right] = 0 \tag{3.30}$$

$$\bar{\rho} = 1 \tag{3.31}$$

$$\frac{\partial p_S}{\partial \tilde{y}} = 0 \tag{3.32}$$

where $\tilde{y} = y/\delta$ and $r_0(x) = x\theta$.

By introducing a stream function defined by

$$\frac{\partial \psi}{\partial y} = ur_0(x), \qquad \psi_x = -vr_0(x) \tag{3.33}$$

we obtain the simplified Bernoulli equation of Section II:

$$\frac{\partial}{\partial x}\left(\frac{u^2}{2}(x,\psi) + p(x,\psi)\right) = 0 \tag{3.34}$$

We also introduce the variable x^*, which marks the location of the origin of the streamline ψ on the surface of the cone:

$$r_0(x^*)dx^* = -\frac{d\psi}{v_w(\psi)} \tag{3.35}$$

The height above the cone surface, $\tilde{y}(x,\psi)$, can be obtained from

$$r_0(x)dy = \frac{d\psi}{u} \tag{3.36}$$

or

$$r_0(x)\tilde{y}(x,x^*) = r_0(x)S(x) - \int_0^{x^*} \frac{r_0(\xi)v_w(\xi)}{u(\xi,x)}\,d\xi \tag{3.37}$$

Introducing the Bernoulli equation, Eq. (3.34), we obtain the height $S(x)$:

$$r_0(x)S(x) = \frac{1}{(2)^{1/2}} \int_0^x \frac{r_0(\xi)v_w(\xi)}{\left(p_S(\xi) - p_S(x)\right)^{1/2}} d\xi \tag{3.38}$$

We now consider a solution to the inverse problem of finding the blowing distribution which leads to a prescribed S(x).

From the Appendix,

$$\frac{r_0(x)v_w(x)}{(2)^{1/2}} = -\frac{1}{\pi}\frac{dp}{dx} \int_0^x \frac{\frac{d}{d\xi}\left(r_0(\xi)S(\xi)\right)}{\left(p_S(\xi) - p_S(x)\right)^{1/2}} d\xi \tag{3.39}$$

An especially simple example of the application of the theory corresponds to

$$p(x) = a_0 - a_1 x \tag{3.40}$$

then

$$S(x) = \frac{a_0}{2} x - \frac{a_1}{5} x^2$$

$$v_w(x) = (a_1 \times 2)^{1/2} \frac{1}{\pi} \left\{ a_0 x^{1/2} \times \frac{4}{3} - a_1 x^{3/2} \times \frac{8}{25} \right\} \tag{3.41}$$

Since $S(1) = 1$, then

$$a_2 = \frac{5}{2} a_1 - 5 > 0$$

IV. COMPRESSIBLE INNER LAYERS

If the limit of the gas density as $\delta \to 0$ is not a constant, we must consider the effect of compressible inner layers. Examples of these are surfaces which, in the absence of injection, have favorable pressure gradients, or situations in which the hypersonic parameter $M\delta \to \infty$. In this section we derive the simplified inviscid equations of motion corresponding to a compressible inner layer. The free stream is assumed to be hypersonic.

The following consistent set of expansions is assumed valid as $\delta \to 0$, with $\tilde{y}$ fixed (it is assumed that $M_\infty^2 \to \infty$, $M_\infty^2 \epsilon(\delta) \to \infty$, and $\delta \to 0$):

$$q_x = \left(\frac{R_0 T_w}{\mu_w}\right)^{1/2} u(x,\tilde{y}) \tag{4.1}$$

$$q_y = \left(\frac{R_0 T_w}{\mu_w}\right)^{1/2} \delta v(x,\tilde{y}) \tag{4.2}$$

$$\frac{T_0}{T_w} = t_0(x,\tilde{y}) \tag{4.3}$$

$$p(x,y) = \rho_\infty U_\infty^2 \epsilon(\delta) p(x,\tilde{y}) \tag{4.4}$$

$$\rho = \frac{\rho_\infty U_\infty^2}{\frac{R_0 T_w}{\mu_w}} \epsilon(\delta)\bar{\rho}(x,\tilde{y}) \tag{4.5}$$

$$\frac{T}{T_w} = t(x,\tilde{y}) \tag{4.6}$$

where x and y are boundary-layer coordinates along and normal to the surface. With these expansions, the momentum equation balances inertial

and pressure-gradient forces, the continuity equation is preserved, and the normal pressure gradient $\partial p/\partial\tilde{y}(x,\tilde{y}) \to 0$ as δ approaches zero.

The momentum equation becomes

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\bar{\rho}}\frac{dp}{dx} \tag{4.7}$$

The continuity equation remains

$$\frac{\partial}{\partial x}(\bar{\rho}u) + \frac{\partial}{\partial\tilde{y}}(\bar{\rho}v) = 0 \tag{4.8}$$

and the energy equation becomes

$$u\frac{\partial t_0}{\partial x} + v\frac{\partial t_0}{\partial y} = 0 \tag{4.9}$$

The total enthalpy, in nondimensional form, is then

$$t_0(x,\tilde{y}) = t(x,\tilde{y}) + \frac{\gamma - 1}{2\gamma}u^2(x,\tilde{y}) \tag{4.10}$$

and the equation of state becomes

$$\bar{\rho} = \frac{\tilde{p}(x,\tilde{y})}{t(x,\tilde{y})} \tag{4.11}$$

Von Mises variables are introduced, replacing $x,\tilde{y}$ by x,ψ. The transformation formulas are

$$\psi_{\tilde{y}} = \bar{\rho}u \tag{4.12a}$$

$$\psi_x = -\bar{\rho}v$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x} + \frac{\partial}{\partial x}\frac{\partial}{\partial \psi} \tag{4.12b}$$

$$\frac{\partial}{\partial \tilde{y}} = \frac{\partial \psi}{\partial \tilde{y}}\frac{\partial}{\partial \psi}$$

The momentum equation becomes

$$u\,\frac{\partial u}{\partial x}\,(u,\psi) + \frac{\partial p}{\bar{\rho}\,\partial x}\,(x,\psi) = 0 \tag{4.13}$$

and the energy equation becomes

$$u\,\frac{\partial t_0}{\partial x} = 0 \tag{4.14}$$

The integral of the total energy equation is then $t_0(x,\psi) = \tilde{t}_0(\psi)$, where $\tilde{t}_0(\psi)$ is a known function of ψ corresponding to the wall temperature. The momentum equation can be integrated, giving:

$$\frac{\gamma}{\gamma - 1}\,\tilde{t}_0(\psi) - \frac{u^2}{2}\,(x,\psi) = K(\psi) \times p(x,\psi)^{\frac{\gamma-1}{\gamma}} \tag{4.15}$$

Changing variables to x,x^*, where x^* marks the location on the surface where the ψ streamline enters the flow, we obtain:

$$\bar{\rho}(x^*,x^*)v_w(x^*)dx^* = -d\psi \tag{4.16}$$

At a point on the surface where $x = x^*$, $u = 0$ and $p = p(x^*,x^*)$. Thus

$$u^2(x,x^*) = 2t_0(x^*,x^*) \times \frac{\gamma}{\gamma - 1}\left\{1 - \left[\frac{p(x,x^*)}{p(x^*,x^*)}\right]^{\frac{\gamma-1}{\gamma}}\right\} \tag{4.17}$$

In order to locate the point x,y corresponding to x,x^*, we use

$$d\tilde{y} = \frac{d\psi}{\bar{\rho}u}$$

$$d\tilde{y} = -\frac{v(x^*,x^*)\bar{\rho}(x^*,x^*)}{\bar{\rho}(x^*,x)u(x^*,x)}\,dx^* \tag{4.18}$$

Noting that

$$\frac{p^{1/\gamma}(x^*,x)}{\bar{\rho}(x^*,x)} = \frac{p^{1/\gamma}(x^*,x^*)}{\bar{\rho}(x^*,x^*)} \tag{4.19}$$

then

$$d\tilde{y} = -v_w(x^*)\left\{\frac{p(x^*,x^*)}{p(x^*,x)}\right\}^{1/\gamma}\frac{dx^*}{u(x^*,x)} \tag{4.20}$$

Equation (4.20) can be integrated to yield

$$\tilde{y}(x^*,x) = S(x) - \int_0^{x^*} v_w(\xi)\left[\frac{p(\xi,\xi)}{p(\xi,x)}\right]^{1/\gamma}\frac{d\xi}{\left(u(\xi,x)\right)^{1/2}} \tag{4.21}$$

Since $\tilde{y}(x,x) = 0$,

$$S(x) = \frac{1}{\left(\frac{2\gamma}{\gamma-1}\right)^{1/2}}\int_0^{x} v(\xi)\left\{\frac{p(\xi,\xi)}{p(\xi,x)}\right\}^{1/\gamma}\frac{d\xi}{\left(t_0(\xi)\left\{1-\left[\frac{p(\xi,x)}{p(\xi,\xi)}\right]^{\frac{\gamma-1}{\gamma}}\right\}\right)^{1/2}} \tag{4.22}$$

<u>Example -- Power-Law Nose, $\varepsilon(\delta) = \delta^2$</u>

Let

$$S = x^n, \qquad n < 1 \tag{4.23}$$

For a power-law effective nose shape, hypersonic small-disturbance theory yields

$$\tilde{p} = A_n \times x^{2(n-1)} \tag{4.24}$$

The injection distribution is $v_w = v_m x^m$; Eq. (4.22) is used to determine v_m and m as follows:

$$\left(\frac{2\gamma}{\gamma - 1}\right)^{1/2} t_0 x^n = v_m \int_0^x \xi^m \left[\frac{\xi}{x}\right]^{\frac{2(n-1)}{\gamma}} \frac{d\xi}{\left(1 - \left(\frac{x}{\xi}\right)^{2(n-1)} \times \frac{\gamma-1}{\gamma}\right)^{1/2}} \tag{4.25}$$

or

$$\left(\frac{2\gamma}{\gamma - 1}\right)^{1/2} t_0 \times x^n = v_m \times x^{m+1} \int_0^1 \frac{\lambda^{m + \frac{2}{\gamma}(n-1)}}{\left(1 - (\lambda)^{-2(n-1)} \times \frac{\gamma-1}{\gamma}\right)^{1/2}} d\lambda \tag{4.26}$$

where $\lambda = \xi/x$. Therefore

$$n = 1 + m \tag{4.27}$$

$$v_m = \frac{\left(\frac{2\gamma}{\gamma - 1}\right)^{1/2} \times t_0}{\int_0^1 \frac{\lambda^A d\lambda}{\left(1 - \lambda^B\right)^{1/2}}} \tag{4.28}$$

where

$$A = m + \frac{2}{\gamma}(n - 1)$$

$$B = -2(n - 1) \times \frac{\gamma - 1}{\gamma}$$

We note that

$$\int_0^1 \frac{\lambda^A d\lambda}{\left(1 - \lambda^B\right)^{1/2}} = \frac{1}{B}\int_0^1 \frac{dx \times x^{\frac{A+1-B}{B}}}{\left(1 - x\right)^{1/2}} \tag{4.29a}$$

$$= \frac{1}{B}\,\frac{\Gamma\left(\frac{A+1}{B}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{A+1}{B} + \frac{1}{2}\right)} \tag{4.29b}$$

where Γ is the gamma function.

<u>Example -- Two-Dimensional Stagnation Point</u>

Although this is not strictly a blowhard problem, it can be treated by the thin-layer inviscid theory as follows:

$$\epsilon(\delta) = 1$$

$$p \sim P_0[1 - ax^2] + \ldots \tag{4.30}$$

$$v_w = \text{constant}$$

$$S(x) = \frac{v_w}{\left(\frac{2\gamma}{\gamma - 1}\right)^{1/2}} \int_0^X \frac{\left[1 - \frac{a}{\gamma}(\xi)^2 + \frac{a}{\gamma}(x)^2\right]}{\left[t_0 a\,\frac{\gamma - 1}{\gamma}\,(x^2 - \xi^2)\right]^{1/2}}\, d\xi \tag{4.31}$$

$$S(x) = \frac{v_w}{(2a)^{1/2}} \int_0^X \frac{1}{(t_0)^{1/2}\left(x^2 - \xi^2\right)^{1/2}}\, d\xi \qquad (4.32)$$

$$= \frac{v_w}{(2)^{1/2}(t_0 a)^{1/2}} \int_0^1 \frac{1}{\left(1 - \lambda^2\right)^{1/2}}\, d\lambda \qquad (4.33)$$

$$= \frac{v_w}{(2)^{1/2}(t_0 a)^{1/2}} \times \frac{\pi}{2} \qquad (4.34)$$

V. COMPRESSIBLE AXISYMMETRIC FLOW

The rotationally symmetric version of the compressible blowhard equations for a hypersonic free stream is derived in this section. A boundary-layer coordinate system is used and an additional parameter R/δ is introduced (see Fig. 3).

$$r = y \cos \theta + Rr_0(x)$$

$$\tilde{r} = \tilde{y} \cos \theta + \frac{R}{\delta} r_0(x)$$

$$q_x = \left(\frac{R_0 T_w}{\mu_w}\right)^{1/2} u(x,\tilde{y}) \tag{5.1}$$

$$q_y = \left(\frac{R_0 T_w}{\mu_w}\right)^{1/2} \delta v(x,\tilde{y}) \tag{5.2}$$

$$\frac{T_0}{T_w} = t_0(x,\tilde{y}) \tag{5.3}$$

$$p(x,y) = \rho_\infty U_\infty^2 \epsilon\left(\delta, \frac{\delta}{R}\right) p(x,\tilde{y}) \tag{5.4}$$

$$\frac{T}{T_w} = t(x,\tilde{y}) \tag{5.5}$$

The inviscid balance between inertia and pressure gradient becomes

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial \tilde{y}} = -\frac{1}{\bar{\rho}} \frac{dp}{dx} \tag{5.6}$$

The continuity equation becomes

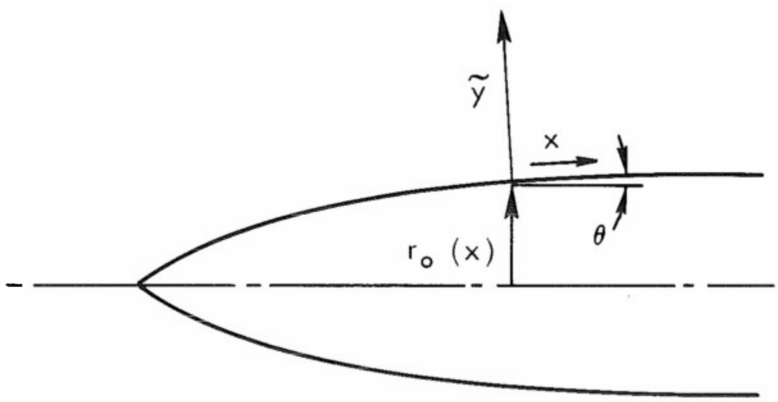


Fig. 3 -- Coordinate system for axisymmetric flow.

$$(u\tilde{r}\bar{\rho})_x + (v\tilde{r}\bar{\rho})_{\tilde{y}} = 0 \tag{5.7}$$

and the energy equation becomes

$$ut_{0x} + vt_{0\tilde{y}} = 0 \tag{5.8}$$

The equation of state is $\bar{\rho} = \bar{p}/t$. The boundary conditions are:

$$\tilde{y} = 0, \qquad v = v_w(x), \qquad u = 0, \qquad t_0 = t_{0w} \tag{5.9}$$

$$\tilde{y} = S, \qquad \frac{v}{u} = \frac{dS}{dx}$$

Introducing the stream-function variables

$$\bar{\rho}u\tilde{r} = \psi_{\tilde{y}} \tag{5.10}$$

$$\bar{\rho}v\tilde{r} = -\psi_x$$

we obtain the compressible Bernoulli equation, which gives us

$$u^2(x^*,x) = 2t_0(x^*) \times \frac{\gamma}{\gamma - 1}\left\{1 - \frac{p(x,x^*)}{p(x^*,x^*)}^{\frac{\gamma-1}{\gamma}}\right\} \tag{5.11}$$

where

$$dx^* = -\frac{d\psi}{\bar{\rho}(x^*,x^*) \times v_w r_0(x^*)} \tag{5.12}$$

In order to locate the point (x^*,x), we must obtain the $\tilde{y}$ coordinate from

$$\tilde{r}d\tilde{y} = \frac{d\psi}{\bar{\rho}u} \tag{5.13}$$

or

$$\frac{\tilde{y}^2}{2}\cos\theta(x) + \frac{R}{\delta}r_0(x)\tilde{y} = \frac{S^2}{2}\cos\theta(x) + r_0 S \times \frac{R}{\delta}$$

$$- \int \frac{r_0 v_w \bar{\rho}(\xi^*,\xi^*)}{\bar{\rho}(\xi,x)u(\xi,x)}\,d\xi \tag{5.14}$$

The edge of the "blown layer" is determined from

$$\frac{S^2}{2}\cos\theta + \frac{R}{\delta}r_0 S = \frac{R}{\delta}\int_0^x \frac{r_0 v_w d\xi}{u(\xi,x)}\left(\frac{p(\xi,\xi)}{p(\xi,x)}\right)^{1/\gamma} \tag{5.15}$$

For practical applications, it is anticipated that $R/\delta \sim O(1)$. Further analysis, involving the introduction of a pressure formula for the outer flow, leads to the formulation of an inverse problem like those in the preceding sections.

VI. CONCLUSIONS

A simple theory has been proposed for the analysis of hypersonic or supersonic flows over cones, flat plates, and wedges, where there is continuous mass injection from the surface. The mass transfer rates are assumed to be low enough to preserve the slenderness of the effective body produced by the injection but high enough to lift the boundary layer off the wall. Since a favorable pressure gradient is necessary to drive the flow in the injection layer, the theory applies only to those injection distributions that cause the pressure distribution to fall. Mass transfer must be provided externally; among the possible methods are: surface ablation caused by externally applied thermal radiation, transpiration through porous walls, or a reaction occurring at the boundary. Mass transfer caused by convective heat transfer is excluded from the theory because the heat transfer rates are zero for high blowing rates.

Additional extensions of the theory might entail (1) calculation of the inviscid flow past the point where the injection has stopped, (2) calculation of the viscous corrections for a finite Reynolds number (which will require a pressure correction due to a viscous displacement thickness), and (3) estimates of aerodynamic coefficients for bodies at small angles of attack.

An unsolved basic problem of the elementary theory is whether solutions exist for constant blowing or constant pressure or both. Presumably this type of flow in two dimensions demands a more accurate theory. One example for flow over a cone with constant blowing is given in Ref. 11. When the assumptions of the present theory are satisfied, the examples discussed in the preceding sections can be compared with the results of other, more elaborate numerical analyses. Comparison of theory with experiment is not yet possible, since the injection distributions of the present work have not yet been simulated in wind-tunnel tests.

Appendix

SOLUTION OF THE INTEGRAL EQUATION (2.30)

The basic integral equation to be solved in the region $x \geq 0$ is (2.30):

$$(2\sigma)^{1/2} S(x) = \int_0^x \frac{v_w(\xi)}{\left(P(\xi) - P(x)\right)^{1/2}} d\xi \qquad \text{(A-1)}$$

Equation (A-1) is transformed into an Abel-type equation by regarding $P(0) - P(x)$ as a new variable. Let

$$\begin{aligned} \mu &= P(0) - P(\xi) \\ \eta &= P(0) - P(x) \end{aligned} \qquad \text{(A-2)}$$

assuming that $P(0)$ is finite and that μ, η are monotone functions. Then

$$(2\sigma)^{1/2} S\left(x(\eta)\right) = \int_0^\eta \frac{v_w\left(\xi(\mu)\right)\left(\frac{d\xi}{d\mu}\right)}{(\eta - \mu)^{1/2}} d\mu \qquad \text{(A-3)}$$

In this form the equation is a standard Abel type (Ref. 10) and has the solution

$$v_w\left(x(\eta)\right) \frac{dx}{d\eta} = \frac{(2\sigma)^{1/2}}{\pi} \frac{d}{d\eta} \int_0^\eta \frac{S\left(\xi(\mu)\right)}{(\eta - \mu)^{1/2}} d\mu \qquad \text{(A-4)}$$

Integration by parts and the use of $S(0) = 0$ thus yields

$$v_w\left(x(\eta)\right) \frac{dx}{d\eta} = \frac{(2\sigma)^{1/2}}{\pi} \int_0^\eta \frac{S'\left(\xi(\mu)\right) \frac{d\xi}{d\mu}}{(\eta - \mu)^{1/2}} d\mu \qquad \text{(A-5)}$$

The reintroduction of the original variables of Eq. (A-2) gives the final result:

$$v_w(x) = - \frac{(2\sigma)^{1/2}}{\pi} \left(\frac{dP}{dx}\right) \int_0^x \frac{S'(\xi)}{\left(P(\xi) - P(x)\right)^{1/2}} d\xi \qquad \text{(A-6)}$$

which is identical to Eq. (2.31).

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